

SuperParticle realization of twisted $\mathcal{N} = 2$ SUSY algebra

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Abstract

We construct a pseudoclassical particle model associated to the twisted $\mathcal{N} = 2$ SUSY algebra in four dimensions. The particle model has four kappa symmetries. Three of them can be used to reduce the model to the vector supersymmetry particle case. The quantization of the model gives rise to two copies of the 4d Dirac equation. The kappa symmetries result to be associated to 4 TSUSY invariant bilinear odd operators who are null operators when a particular condition is satisfied. These operators are in correspondence one to one with analogous operators existing in the case of the $\mathcal{N} = 2$ SUSY algebra, making both cases 1/2 BPS.

1 Introduction

Supersymmetry plays a crucial role in field theories, supergravities and String/M theory. In flat space-time supersymmetry is characterized by the presence of odd spinor charges that together with the generators of the Poincaré group form the target space Super Poincaré group.

There is also an odd vector extension of the Poincaré group, called the Vector Super Poincaré (VSUSY) [1]. Witten [2] introduced topological $\mathcal{N} = 2$ Yang-Mills theories by performing a topological twist in euclidean 4 dimensional space-time. After this twist, the fermionic generators become a vector, a scalar and an anti-selfdual tensor [3, 4]. After truncation of the anti-selfdual sector, the twisted supersymmetry (TSUSY) algebra coincides with the euclidean VSUSY algebra.

A pseudoclassical particle model invariant under Super Poincaré was introduced in [5] and extended later to ten dimensions [6]. In this last case the lagrangian consists of the Nambu-Goto and the Wess-Zumino terms. Choosing a particular set of values of their coefficients, a fermionic gauge symmetry called kappa symmetry [7] [8] appears. In the case of $\mathcal{N} = 2$ Super Poincaré in four dimensions (4d) the number of independent kappa symmetries is four and therefore the superparticle is 1/2 BPS. The covariant quantization of this model has several difficulties.

A pseudoclassical particle invariant under VSUSY Poincaré is the spinning particle introduced in [1]. The quantization preserving VSUSY gives rise to two copies of the 4d Dirac equation [9]. On the other hand, breaking the rigid supersymmetry by a suitable constraint on the Grassmann variables, one recovers the 4d Dirac equation of [1, 11, 10, 12]. The lagrangian contains a Dirac-Nambu-Goto piece and two Wess-Zumino terms. For particular values of the coefficients of the lagrangian, the model has world line gauge supersymmetry which is analogous to the fermionic kappa symmetry of the superparticle case. When we require world-line supersymmetry the model has bosonic BPS configurations that preserve 1/5 of the vector supersymmetry. The fact that the model is 1/5 BPS makes it difficult to find a relation with the superparticle, since this is 1/2 BPS.

In this paper we construct a pseudoclassical particle model in a 4d euclidean space which is invariant under the twisted $\mathcal{N} = 2$ SUSY algebra. The variables of lagrangian are the space-time coordinates x^μ and the odd real Grassmann variables $\xi^\mu, \xi^5, \xi^{\mu\nu}$ that are a vector, a scalar and a self-dual tensor under 4d rotations. The lagrangian contains a Dirac-Nambu-Goto piece and two Wess-Zumino terms. For particular values of the coefficients of the lagrangian, the model has *four* fermionic kappa symmetries. Three of them allow to eliminate the selfdual Grassmann variables $\xi^{\mu\nu}$ and the result is equivalent to the euclidean spinning particle model invariant under VSUSY. The quantization of the model in the reduced space leads to two copies of the 4d euclidean Dirac equation.

The model has bosonic BPS configurations that preserve 1/2 of the twisted $\mathcal{N} = 2$ supersymmetry. Note that the TSUSY model is 1/2 BPS like the superparticle. This is expected on the basis of the correspondence of the TSUSY and $\mathcal{N} = 2$ SUSY algebras and we will show explicitly this relation.

2 TSUSY algebra

We will consider the SUSY algebra for $\mathcal{N} = 2$ in the euclidean case. The algebra of the odd generators is given by

$$\{Q_\alpha^i, \tilde{Q}_{jA}\} = 2(\sigma^\mu)_{\alpha A} P_\mu \delta_j^i, \quad \{Q_\alpha^i, Q_\beta^j\} = \epsilon_{\alpha\beta} \epsilon^{ij} Z_1, \quad \{\tilde{Q}_{iA}, \tilde{Q}_{jB}\} = \epsilon_{AB} \epsilon_{ij} Z_2 \quad (1)$$

where $\sigma^\mu = (i\tau_j, 1)$ and $\epsilon = i\tau_2$ and τ_i are the Pauli matrices. The indices i, α, A describe the following groups

$$i \in SU(2)^\mathcal{R}, \quad \alpha \in SU(2)_L, \quad A \in SU(2)_R, \quad (2)$$

where $SU(2)^\mathcal{R}$ is the R -symmetry group and $SU(2)_L \times SU(2)_R$ describes the space-time rotation group in four dimensions, $O(4)$. This algebra can be twisted in the following way [2], see also [3, 4]. We identify a new rotation group $O(4)_N$ with the product of the diagonal part of the product of $SU(2)^\mathcal{R} \times SU(2)_L$ times $SU(2)_R$. That is

$$O(4)_N \approx [SU(2)^\mathcal{R} \times SU(2)_L]_D \times SU(2)_R = SU(2)_D \times SU(2)_R. \quad (3)$$

Where we have denoted by $SU(2)_D$ the diagonal subgroup of the direct product in parenthesis in the previous equation. In other words we identify the spinor indices of the internal symmetry $SU(2)^\mathcal{R}$ with the spinor indices of the space-time symmetry $SU(2)_L$. Then, the indices i, α transform both under $SU(2)_D$, whereas A under $SU(2)_R$. We have

$$Q_\alpha^i \in (1/2, 0) \times (1/2, 0) = (0, 0) \oplus (1, 0), \quad \tilde{Q}_{iA} \in (1/2, 0) \times (0, 1/2) = (1/2, 1/2). \quad (4)$$

It is important to notice that the two inequivalent spinorial representations in $O(4)$ (from now on $O(4)_N \rightarrow O(4)$) are both pseudoreal. We see that Q_α^i describes a scalar and a self-dual tensor, that is a spin 1, whereas \tilde{Q}_{iA} a fourvector. This can be made explicit by decomposing the supercharges Q and \tilde{Q} as follows¹

$$Q = iG_5 - \frac{i}{2}\sigma^{\mu\nu}G_{\mu\nu}, \quad \tilde{Q} = i\sigma^\mu G_\mu. \quad (5)$$

In terms of $(G_\mu, G_5, G_{\mu\nu})$ the twisted algebra is given by

$$\{G_5, G_5\} = \tilde{Z}, \quad \{G_5, G_\mu\} = -P_\mu, \quad \{G_\mu, G_\nu\} = Z\delta_{\mu\nu} \quad (6)$$

$$\{G_{\mu\nu}, G_5\} = 0, \quad \{G_{\mu\nu}, G_{\rho\lambda}\} = 4P_{\mu\nu, \rho\lambda}^+ \tilde{Z}, \quad \{G_{\mu\nu}, G_\rho\} = -4P_{\mu\nu, \rho\lambda}^+ P^\lambda \quad (7)$$

$$[G_\mu, M_{\rho\sigma}] = -i\eta_{\mu[\rho} G_{\sigma]}, \quad [G_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\nu[\rho} G_{\mu\sigma]} + i\eta_{\mu[\rho} G_{\nu\sigma]}, \quad [G_5, M_{\rho\sigma}] = 0. \quad (8)$$

Here

$$\tilde{Z} = -\frac{Z_1}{2}, \quad Z = -\frac{Z_2}{2}, \quad (9)$$

and $P_{\mu\nu, \rho\sigma}^\pm$ project out the (anti)self-dual part of a tensor:

$$P_{\mu\nu, \rho\lambda}^\pm = \frac{1}{4}(\delta_{\mu\rho}\delta_{\nu\lambda} - \delta_{\nu\rho}\delta_{\mu\lambda} \pm \epsilon_{\mu\nu\rho\lambda}). \quad (10)$$

¹For simplicity we suppress the indices whenever it does not create ambiguities.

Notice that due to the self-dual properties of $\sigma_{\mu\nu}$, $G_{\mu\nu}$ is self-dual with three independent components satisfying $G_{\mu\nu} = P_{\mu\nu,\rho\lambda}^+ G^{\rho\lambda}$.

In Minkowski space SUSY algebra admits a $U(1)$ automorphism group generated by the ghost number. In the euclidean case such an automorphism does not exist due to the pseudo-reality of the spinors. On the other hand there is an automorphism given by the following scale transformation (we will call this ghost dilation to be distinguished from the automorphism generated by the usual dimensional scale dilations). This is given by

$$\begin{aligned} P_\mu &\rightarrow P_\mu, & M_{\mu\nu} &\rightarrow M_{\mu\nu}, & Z_1 &\rightarrow e^{+2\lambda} Z_1, & Z_2 &\rightarrow e^{-2\lambda} Z_2 \\ Q_\alpha^i &\rightarrow e^{+\lambda} Q_\alpha^i, & \tilde{Q}_{iA} &\rightarrow e^{-\lambda} \tilde{Q}_{iA}. \end{aligned} \quad (11)$$

Clearly this symmetry of the algebra is related to the ghost symmetry through the analytic continuation $\lambda \rightarrow i\lambda$. The corresponding transformation properties of the scalar, vector and tensor generators are

$$G_5 \rightarrow e^{+\lambda} G_5, \quad G_\mu \rightarrow e^{-\lambda} G_\mu, \quad G_{\mu\nu} \rightarrow e^{+\lambda} G_{\mu\nu}. \quad (12)$$

Notice that, in Minkowski space the BPS operators are constructed (in the rest frame) by taking linear combinations of Q and Q^\dagger . However, due to the pseudo-reality condition, in the euclidean case one has to take linear combinations of Q and \tilde{Q} . The total number of BPS operators in TSUSY is expected to be 4 as in $\mathcal{N} = 2$ SUSY. However, since the BPS charges should be expressed in terms of the generators G_5, G_μ and $G_{\mu\nu}$, it is easier to work in an arbitrary Lorentz frame. We should be able to construct 4 linear combinations, one involving G_5 and G_μ and the other G_μ and $G_{\mu\nu}$. This is because G_5 and $G_{\mu\nu}$ depend on Q whereas G_μ depends on \tilde{Q} . We then construct the following four combinations for $P^2 \neq 0$

$$B = P^\mu G_\mu + Z G_5, \quad B_\mu = G_{\mu\nu} P^\nu + \tilde{Z} P_{\mu\nu}^\perp, \quad (13)$$

where

$$P_{\mu\nu}^\perp = \delta_{\mu\nu} - \frac{P_\mu P_\nu}{P^2}. \quad (14)$$

These operators are Lorentz covariant and scale correctly under the dilation automorphism discussed previously. Notice that the transverse projector ensures that only three operators out of B_μ are independent. Let us now evaluate the anticommutators of these operators. We get

$$\{B, B\} = -(P^2 - Z\tilde{Z})Z, \quad \{B_\mu, B_\nu\} = -(P^2 - Z\tilde{Z})\tilde{Z}P_{\mu\nu}^\perp, \quad \{B, B_\mu\} = 0. \quad (15)$$

Let us also evaluate the anticommutators with the odd generators of TSUSY. We find

$$\{G_\mu, B\} = \{G_{\mu\nu}, B\} = 0, \quad \{G_5, B\} = -(P^2 - Z\tilde{Z}), \quad (16)$$

$$\{G_\mu, B_\nu\} = -(P^2 - Z\tilde{Z})P_{\mu\nu}^\perp, \quad \{G_5, B_\mu\} = \{G_{\mu\nu}, B_\rho\} = 0. \quad (17)$$

Therefore, when the condition

$$P^2 = Z\tilde{Z} \quad (18)$$

is satisfied, the operators B and the three independent components of B_μ are four null operators invariant under TSUSY transformations. The states annihilated by the operators B, B_μ are the BPS states. In the rest frame one gets

$$B = -\frac{i}{2} \left(m \text{Tr}(\tilde{Q}) + Z \text{Tr}(Q) \right), \quad (19)$$

$$B_4 = 0, \quad B_i = \frac{1}{2} \left(m \text{Tr}(\tau_i Q) - \tilde{Z} \text{Tr}(\tau_i \tilde{Q}) \right). \quad (20)$$

From these expressions one recovers easily the usual expressions for the BPS operators in the language of SUSY except for the factors m and Z , \tilde{Z} . In fact, in Minkowski space, in order to get BPS operators with definite ghost number it is enough to take the hermitian conjugate of Q . Since the $U(1)$ generated by the ghost number goes into dilation group, one needs to insert appropriate factors Z , \tilde{Z} , since Q and \tilde{Q} scale in a different way. However, Z and \tilde{Z} bring dimensions requiring the insertion of appropriate powers of mass.

When the condition (18) is satisfied we can evaluate both G_5 and $G_{\mu\nu}$ using the equations $B = B_\mu = 0$. We get

$$G_5 = -\frac{1}{Z} P_\mu G^\mu, \quad G_{\mu\nu} = \frac{4}{Z} P_{\mu\nu,\rho\lambda}^+ P^\rho G^\lambda. \quad (21)$$

In the rest frame we have

$$G_5 = -\frac{m}{Z} G_4, \quad G_{i4} = -\frac{m}{Z} G_i \quad (22)$$

with the further duality condition

$$G_{ij} = \epsilon_{ijk} G_{k4}. \quad (23)$$

In the next section we will study the issue of realizing the TSUSY algebra on the phase space of a single particle. In principle, the particle model is described by the position variables and by 8 real Grassmann variables associated to the odd generators of TSUSY, ξ_5 , ξ_μ and $\xi_{\mu\nu}$, where the last variables form a self-dual tensor. If one chooses the parameters of the action in such a way that the realization of the condition (18) is satisfied, one expects that the number of the Grassmann variables describing the model can be reduced. This is because B and B_μ become null operators. In the particle model the corresponding expressions will vanish giving rise to constraints. On the other hand, the equations (15) show that these operators anticommute with each other. As a consequence, at the level of the particle model they give rise to 4 first-class constraints. As it is well known, first class constraints generate gauge transformations (kappa-symmetries in the case of SUSY) and this allows to eliminate one odd variable in configuration space for each transformation via gauge fixing. For example, one can fix three gauge conditions and eliminate the variables $\xi_{\mu\nu}$. In this case one obtains the VSUSY particle model of reference [9]. As a further step it is possible to use the remaining constraint to eliminate ξ_5 . The possibility of eliminating the variables ξ_5 and $\xi_{\mu\nu}$ goes together with the fact that, when the condition (18) is satisfied, the generators G_5 and $G_{\mu\nu}$ can be expressed through G_μ (see eq. (21)). In the euclidean case, this means that the model involving only ξ_μ has the full TSUSY invariance. However, in Minkowski space this does not happen because the self-dual generators $G_{\mu\nu}$, and the corresponding parameters $\xi_{\mu\nu}$, lose the reality property.

The mechanism discussed here looks completely general, that is, when the choice of a particular realization of the algebra gives rise to null operators, in the associated particle model there are constraints. The character of these constraints (first or second class) will depend on the algebraic relations existing among the null operators. In the present case, since the 4 null operators anticommute among themselves, they give rise to 4 first-class constraints.

3 TSUSY particle

To construct a pseudoclassical model invariant under TSUSY we will use the method of non-linear realization [13]. The starting point is to consider the Maurer-Cartan (MC) form associated to the twisted euclidean superspace, $\frac{G}{H} = \frac{\text{TSUSY}}{SO(4)}$ which locally is parametrized by

$$g_0 = e^{iP_\mu x^\mu} e^{\frac{i}{4}G_{\mu\nu}\xi^{\mu\nu}} e^{iG_5\xi^5} e^{iG_\mu\xi^\mu} e^{iZc} e^{i\tilde{Z}\tilde{c}}. \quad (24)$$

The MC 1-form of the twisted superspace is

$$\Omega_0 = -ig_0^{-1}dg_0 = P_\mu \tilde{L}_P^\mu + G_5 \tilde{L}_G^5 + G_\mu \tilde{L}_G^\mu + \frac{1}{4}G_{\mu\nu} \tilde{L}_G^{\mu\nu} + Z \tilde{L}_Z + \tilde{Z} \tilde{L}_{\tilde{Z}}. \quad (25)$$

The even components of the 1-form are given by

$$\begin{aligned} \tilde{L}_P^\mu &= dx^\mu - i\xi^\mu d\xi^5 + i\xi_\sigma d\xi^{\mu\sigma} \\ \tilde{L}_Z &= dc + \frac{i}{2}\xi^\mu d\xi_\mu, \quad \tilde{L}_{\tilde{Z}} = d\tilde{c} + \frac{i}{2}\xi^5 d\xi^5 + \frac{i}{8}\xi_{\mu\nu} d\xi^{\mu\nu} \end{aligned} \quad (26)$$

whereas the odd ones are

$$\tilde{L}_G^\mu = d\xi^\mu, \quad \tilde{L}_G^5 = d\xi^5, \quad \tilde{L}_G^{\mu\nu} = d\xi^{\mu\nu}. \quad (27)$$

The TSUSY transformations of the real superspace coordinates are given by

$$\begin{aligned} G_{\mu\nu} &: \delta\xi^{\mu\nu} = \beta^{\mu\nu}, \quad \delta\tilde{c} = \frac{i}{8}\xi_{\mu\nu}\beta^{\mu\nu}, \\ G_5 &: \delta\xi^5 = \beta^5, \quad \delta\tilde{c} = \frac{i}{2}\xi^5\beta^5, \\ G_\mu &: \delta\xi^\mu = \beta^\mu, \quad \delta x^\mu = -i\xi^5\beta^\mu + i\xi^{\mu\nu}\beta_\nu, \quad \delta c = \frac{i}{2}\xi_\mu\beta^\mu, \\ P_\mu &: \delta x^\mu = \epsilon^\mu, \\ Z &: \delta c = \epsilon_Z, \\ \tilde{Z} &: \delta\tilde{c} = \epsilon_{\tilde{Z}}. \end{aligned} \quad (28)$$

These transformations leave invariant the above MC 1-forms.

Now we want to consider the motion of a massive particle in this space. Then, the natural coset is $\frac{G}{H} = \frac{\text{TSUSY}}{SO(3)}$, regarding x^4 as "euclidean time". Since the "euclidean boosts", M_{4i} are broken spontaneously by the presence of the particle, see for example [14, 15] for the case of relativistic particles with Minkowski signature. Then the elements of the coset are of the form

$$g = g_0 U, \quad U \equiv e^{iM_{4i}v^i}, \quad (29)$$

where v^i are the parameters associated to the "euclidean boost". The corresponding MC form is

$$\begin{aligned} \Omega &= -ig^{-1}dg = U^{-1}\Omega_0 U + U^{-1}dU \\ &= P_\mu L_P^\mu + \frac{1}{2}M_{\mu\nu}L_M^{\mu\nu} + G_5 L_G^5 + G_\mu L_G^\mu + \frac{1}{4}G_{\mu\nu}L_G^{\mu\nu} + Z L_Z + \tilde{Z} L_{\tilde{Z}}, \end{aligned} \quad (30)$$

with

$$L_P^\mu = \Lambda^\mu{}_\nu \tilde{L}_P^\nu, \quad L_M^{\mu\nu} = \Lambda^\mu{}_\rho d\Lambda^\nu{}_\sigma \eta^{\rho\sigma}, \quad L_Z = \tilde{L}_Z, \quad L_{\tilde{Z}} = \tilde{L}_{\tilde{Z}}, \quad (31)$$

$$L_G^\mu = \Lambda^\mu{}_\nu \tilde{L}_G^\nu, \quad L_G^5 = \tilde{L}_G^5, \quad L_G^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \tilde{L}_G^{\rho\sigma}. \quad (32)$$

where we have used

$$U^{-1} P_\nu U = P_\mu \Lambda^\mu{}_\nu(v), \quad U^{-1} M_{\mu\nu} U = M_{\rho\sigma} \Lambda^\rho{}_\mu(v) \Lambda^\sigma{}_\nu(v). \quad (33)$$

$\Lambda^\mu{}_\nu(v)$ is a finite 4d rotation matrix depending on the euclidean boost parameters v^i .

There are three even 1-forms invariant under the left transformations, L_P^4 , L_Z , $L_{\tilde{Z}}$. Then, the simplest form of the particle lagrangian is a linear combination of these invariant one forms:

$$\begin{aligned} \mathcal{L} d\tau &= -[m L_P^4 + \beta L_Z + \gamma L_{\tilde{Z}}]^\sim \\ &= -\left(m \Lambda^4{}_\nu (\dot{x}^\nu - i\xi^\nu \xi^5 + i\xi_\sigma \dot{\xi}^{\nu\sigma}) + \beta (\dot{c} + \frac{i}{2} \xi^\mu \dot{\xi}_\mu) + \gamma (\dot{\bar{c}} + \frac{i}{2} \xi^5 \dot{\xi}^5 + \frac{i}{8} \xi_{\mu\nu} \dot{\xi}^{\mu\nu}) \right) d\tau, \end{aligned} \quad (34)$$

where the $[\dots]^\sim$ denotes the pullback to the world line forms : $[dx^\mu]^\sim \equiv \dot{x}^\mu(\tau) d\tau$, etc. If we introduce the momenta $p_\mu = -m \Lambda^4{}_\mu$ we can write the canonical lagrangian as

$$\mathcal{L}_1 = p_\nu (\dot{x}^\nu - i\xi^\nu \xi^5 + i\xi_\sigma \dot{\xi}^{\nu\sigma}) - \frac{e}{2} (p^2 - m^2) - \beta \frac{i}{2} \xi^\mu \dot{\xi}_\mu - \gamma \left(\frac{i}{2} \xi^5 \dot{\xi}^5 + \frac{i}{8} \xi_{\mu\nu} \dot{\xi}^{\mu\nu} \right), \quad (35)$$

where we have removed total derivative terms.

4 Constraints and Kappa Symmetries of TSUSY Lagrangian

In this section we will compute the constraints and the generators of the gauge symmetries for the TSUSY particle. The momenta for the Lagrangian (34) are given by²

$$p_\mu = -m \Lambda^4{}_\mu, \quad p_{v^i} = 0, \quad (36)$$

$$p_c = -\beta, \quad p_{\bar{c}} = -\gamma, \quad (37)$$

$$\zeta_\mu = -\beta \frac{i}{2} \xi_\mu, \quad (38)$$

$$\zeta_5 = -ip_\mu \xi^\mu - \gamma \frac{i}{2} \xi^5, \quad (39)$$

$$\zeta_{\mu\nu} = 4i p_\rho \xi_\sigma P^{\rho\sigma}{}_{,\mu\nu} - \gamma \frac{i}{2} \xi_{\mu\nu}, \quad (40)$$

where the basic Poisson brackets of the fermionic variables are

$$\{\xi^\mu, \zeta_\nu\} = \delta^\mu{}_\nu, \quad \{\xi^{\mu\nu}, \zeta_{\rho\sigma}\} = 4 P^{\mu\nu}{}_{,\rho\sigma}, \quad \{\xi^5, \zeta_5\} = 1. \quad (41)$$

²the momenta of the fermionic variables are computed using right derivative to define odd momenta then $H = \zeta \dot{\xi} - L$ and $\{\xi, \zeta\} = +1$.

We have seven even constraints (36) , one first class

$$\phi = \frac{1}{2}(p^2 - m^2), \quad (42)$$

and six second class that can be used to eliminate (v^i, p_{v^i}) ,

$$p_{v^i} = 0, \quad v^i = v^i(p_\mu). \quad (43)$$

There are also two even constraints (37) that are first class generating local shift of c and \tilde{c} ,

$$\chi_c = p_c + \beta = 0, \quad \chi_{\tilde{c}} = p_{\tilde{c}} + \gamma = 0. \quad (44)$$

We have four fermionic constraints (38) which are second class

$$\chi_\mu = \zeta_\mu + \beta \frac{i}{2} \xi_\mu = 0. \quad (45)$$

For second class constraints $\phi_\alpha = 0$ we can either define the Dirac bracket

$$\{A, B\}^* = \{A, B\} - \{A, \phi_\alpha\} c_{\alpha\beta}^{-1} \{\phi_\beta, B\} \quad (46)$$

or equivalently introduce the A^* variables associated to any A , as, see for example [16]

$$A^* = A - \{A, \phi_\alpha\} c_{\alpha\beta}^{-1} \phi_\beta \quad (47)$$

where $\{\phi_\alpha, \phi_\beta\} = c_{\alpha\beta}$. The relations (39) and (40) give the odd constraints

$$\chi_5 = \zeta_5 + i p_\mu \xi^\mu + \gamma \frac{i}{2} \xi^5 = 0, \quad (48)$$

$$\chi_{\mu\nu} = \zeta_{\mu\nu} - 4 i p_\rho \xi_\sigma P^{\rho\sigma}{}_{\mu\nu} + \gamma \frac{i}{2} \xi_{\mu\nu} = 0 \quad (49)$$

for which $\chi_5^*, \chi_{\mu\nu}^*$ are

$$\chi_5^* = \zeta_5 + \gamma \frac{i}{2} \xi^5 - \frac{1}{\beta} p_\mu (\zeta_\mu - \beta \frac{i}{2} \xi_\mu) = 0, \quad (50)$$

$$\chi_{\mu\nu}^* = \zeta_{\mu\nu} + \gamma \frac{i}{2} \xi_{\mu\nu} + \frac{4}{\beta} p_\rho (\zeta_\sigma - \beta \frac{i}{2} \xi_\sigma) P^{\rho\sigma}{}_{\mu\nu} = 0. \quad (51)$$

They are first class constraints when the condition

$$m^2 = \beta\gamma \quad (52)$$

is satisfied. In fact, they satisfy the following Poisson brackets

$$\{\chi_5^*, \chi_5^*\} = -\frac{i}{\beta} (p^2 - \beta\gamma) = -\frac{i}{\beta} (2\phi + m^2 - \beta\gamma), \quad (53)$$

$$\{\chi_5^*, \chi_{\mu\nu}^*\} = \frac{4i}{\beta} p_\lambda p_\rho P^{\rho\lambda}{}_{,\mu\nu} = 0,$$

$$\{\chi_{\mu\nu}^*, \chi_{\rho\sigma}^*\} = -4 \frac{i}{\beta} (2\phi + m^2 - \beta\gamma) P_{\mu\nu, \rho\sigma}. \quad (54)$$

Notice that $\chi_{\mu\nu}$ is a self-dual tensor, $P_{\mu\nu, \rho\lambda}^- \chi^{\rho\lambda} = 0$. In total the number of independent kappa symmetries is four like in the case of the $\mathcal{N} = 2$ superparticle, as expected from the discussion made at the end of Section 2.

The kappa transformations are generated by

$$G = \chi_5^* \kappa^5 + \frac{1}{4} \chi_{\mu\nu}^* \kappa^{\mu\nu}, \quad (55)$$

from which

$$\delta \xi^5 = \kappa^5, \quad \delta \xi^{\mu\nu} = \kappa^{\mu\nu}, \quad \delta \xi^\mu = \frac{1}{\beta} (p^\mu \kappa^5 + p_\nu \kappa^{\nu\mu}), \quad \delta x^\mu = i \xi^\mu \kappa^5 + i \xi_\nu \kappa^{\nu\mu} \quad (56)$$

The lagrangian (34) transforms as a total divergence when (52) holds.

The canonical generators of the TSUSY transformations (28) are

$$\begin{aligned} P_\mu &= p_\mu, & Z &= p_c, & \tilde{Z} &= p_{\tilde{c}}, & G_{\mu\nu} &= \zeta_{\mu\nu} + p_{\tilde{c}} \frac{i}{2} \xi_{\mu\nu}, \\ G_5 &= \zeta_5 + p_{\tilde{c}} \frac{i}{2} \xi^5, & G_\mu &= \zeta_\mu - i \xi^5 p_\mu + i p^\nu \xi_{\nu\mu} + p_c \frac{i}{2} \xi_\mu. \end{aligned} \quad (57)$$

In terms of the canonical variables the quantities B and B_μ of equation (13) are given by

$$\begin{aligned} B &= P^\mu G_\mu + Z G_5 = \\ &= \chi_c \frac{i}{2} p^\mu \xi_\mu + \chi_c (\zeta_5 + p_{\tilde{c}} \frac{i}{2} \xi^5) - \beta \chi_5^* - i \xi^5 (p^2 - \beta \gamma), \end{aligned} \quad (58)$$

$$\begin{aligned} B_\mu &= G_{\mu\nu} P^\nu + \tilde{Z} P_{\mu\nu}^\perp G^\nu = \\ &= \chi_{\tilde{c}} \frac{i}{2} \xi_{\mu\nu} p^\nu + \chi_{\tilde{c}} P_{\mu}^{\perp\nu} (\zeta_\nu + i p^\rho \xi_{\rho\nu} + (\chi_c - \beta) \frac{i}{2} \xi_\nu) - \chi_c \gamma P_{\mu}^{\perp\nu} \frac{i}{2} \xi_\nu + \\ &\quad + \chi_{\mu\nu}^* p^\nu + \frac{1}{\beta} P_{\mu}^{\perp\nu} (\zeta_\nu - \beta \frac{i}{2} \xi_\nu) (p^2 - \beta \gamma). \end{aligned} \quad (59)$$

Then, if we use the condition $P^2 - Z \tilde{Z} = p^2 - \beta \gamma = 0$ the B, B_μ are linear combinations of the first class constraints. Their expressions, barring the trivial constraints $\chi_c = \chi_{\tilde{c}} = 0$, are

$$B = -\beta \chi_5^*, \quad B_\mu = \chi_{\mu\nu}^* p^\nu. \quad (60)$$

5 Gauge fixed lagrangian and quantization on the reduced space

Taking into account the first class constraints we have obtained we can introduce the gauge fixing conditions

$$\xi^5 = \xi^{\mu\nu} = c = \tilde{c} = 0. \quad (61)$$

The gauge fixed Lagrangian of (34) is

$$\mathcal{L}^{gf} = -m \Lambda^4_\mu \dot{x}^\mu - \beta \frac{i}{2} \xi^\mu \dot{\xi}_\mu = p_\mu \dot{x}^\mu - \frac{e}{2} (p^2 - m^2) - \beta \frac{i}{2} \xi^\mu \dot{\xi}_\mu. \quad (62)$$

The global symmetry generators are, including compensating gauge transformations,

$$\begin{aligned} G_\mu^* &= \zeta_\mu - \beta \frac{i}{2} \xi_\mu, & G_5^* &= \frac{1}{\beta} p^\mu (\zeta_\mu - \beta \frac{i}{2} \xi_\mu), \\ G_{\mu\nu}^* &= -\frac{4}{\beta} p_\rho (\zeta_\sigma - \beta \frac{i}{2} \xi_\sigma) P^{\rho\sigma}_{,\mu\nu}. \end{aligned} \quad (63)$$

In the Lagrangians (62) the self-dual real coordinates $\xi^{\mu\nu}$ have disappeared thoroughly. Now we can use the lagrangian (62) in Minkowski space, by replacing $m^2 \rightarrow -m^2$. However, the global $G_{\mu\nu}^*$ transformations (63), realized in the Euclidian metric, are not symmetries in Minkowski space due to the complex nature of the self-dual tensor transformation parameters $\epsilon^{\mu\nu}$.

Going to the Minkowski space and fixing the gauge in relation to the reparametrization invariance

$$x^0 = \tau. \quad (64)$$

we can solve the constraint (42) for p_0

$$p_0 = \pm \sqrt{\vec{p}^2 + m^2}. \quad (65)$$

The canonical form of the lagrangian (62) in the reduced space³ becomes

$$L^{C*} = \pm \sqrt{\vec{p}^2 + m^2} + \vec{p} \cdot \dot{\vec{x}} - \beta \frac{i}{2} \xi_\mu \dot{\xi}^\mu. \quad (66)$$

Now we quantize the model. The basic canonical (anti-)commutators are

$$[x^i, p_j] = i\delta^i_j, \quad [\xi^\mu, \xi^\nu]_+ = -\frac{1}{\beta} \eta^{\mu\nu}. \quad (67)$$

Notice that the ambiguity in the sign the energy must be taken into account in the quantum theory. The hamiltonian for the lagrangian (66) is the operator

$$P_0 = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}, \quad \omega \equiv \sqrt{\vec{p}^2 + m^2}, \quad (68)$$

with eigenvalues $\pm\omega$. The Schrödinger equation becomes

$$i\partial_\tau \Psi(\vec{x}, \tau) = P_0 \Psi(\vec{x}, \tau), \quad \Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}, \quad (69)$$

where Ψ_+ and Ψ_- are positive and negative energy states. The odd real variables ξ^μ in (66) must commute with all the bosonic variables, in particular with the energy P_0 in (68). Therefore they must be realized in terms of 8-dimensional gamma matrices

$$\xi^\mu = \sqrt{\frac{-1}{2\beta}} \begin{pmatrix} \gamma^\mu & 0 \\ 0 & -\gamma^\mu \end{pmatrix} \begin{pmatrix} \gamma^5 & 0 \\ 0 & -\gamma^5 \end{pmatrix}, \quad (70)$$

where the γ^μ and γ^5 are the ordinary 4-component gamma matrices in 4-dimensions and Ψ is an 8 component wave function. In [9] we have shown that the equations of motion are equivalent to two set of 4-component Dirac equations using an inverse Foldy-Whouthuysen transformation.

6 BPS Configurations

Here we will consider the BPS equations for the TSUSY particle. The corresponding bosonic supersymmetric configurations appear only when the lagrangians have the

³ An analogous discussion for the lagrangian of spinning particle in [1] was done in reference [9].

four kappa symmetries, i.e., when $m^2 = \beta\gamma$. Now we look for supersymmetric bosonic configurations. For consistency we look for transformations of the fermionic variables not changing their initial value that is supposed to vanish

$$0 = \delta\xi^5|_{\text{fermions}=0} = \epsilon^5 + \kappa^5, \quad (71)$$

$$0 = \delta\xi^\mu|_{\text{fermions}=0} = \epsilon^\mu + \frac{1}{\beta}(p^\mu \kappa^5 + p_\nu \kappa^{\nu\mu}), \quad (72)$$

$$0 = \delta\xi^{\mu\nu}|_{\text{fermions}=0} = \epsilon^{\mu\nu} + \kappa^{\mu\nu}. \quad (73)$$

Notice that all ϵ^μ can be expressed in terms of ϵ^5 and the three independent components of $\epsilon^{\mu\nu}$. The consistency of equation (72) implies

$$p^\mu = \text{constant} \quad (74)$$

which is a 1/2 BPS configuration. Notice that the BPS solutions satisfy the Euler-Lagrange equations of motion.

7 Discussions

In this paper we have considered the twisted version of $\mathcal{N} = 2$ SUSY. The corresponding algebra (TSUSY) contains odd tensorial generators, a scalar, a fourvector and a self-dual tensor. We have shown that there are four odd quadratic expressions in the generators, invariant under odd supertranslations, being null operators when the condition (18) is satisfied. These operators are one scalar, B , and one fourvector, B_μ , with only three independent components (since it is orthogonal to the four-momentum). Requiring the vanishing of the last three independent operators, the TSUSY algebra reduces, as the independent generators are concerned, to the algebra of VSUSY (see eq. [9]).

By using the usual methods of the non linear realizations we have constructed the particle model associated to the TSUSY algebra (still remaining in an euclidean space). The natural variables describing the model are the parameters of the coset space $\frac{\text{TSUSY}}{\text{SO}(4)}$, that is the position x_μ and the odd real quantities ξ_5 , ξ_μ and the self-dual tensor $\xi_{\mu\nu}$. The most simple invariant lagrangian turns out to depend on 3 parameters. The condition leading to four null quadratic operators in the generators of the algebra is equivalent to a condition among these three parameters. When this condition is satisfied, the model acquires four first-class constraints (corresponding to the vanishing of the operators B and B_μ). As a consequence there are four local symmetries, "kappa"-symmetries, allowing to eliminate the variables ξ_5 and $\xi_{\mu\nu}$. This process is just in correspondence with the analogous procedure that in the case of $\mathcal{N} = 2$ SUSY, allows the elimination of four fermionic variables. After this reduction, the euclidean model still has the full TSUSY invariance (with the generators G_5 and $G_{\mu\nu}$ expressed non-linearly in terms of G_μ). At this stage we can also continue the model to the Minkowski space, however the complete TSUSY invariance is lost, due to the complex nature of the self-dual transformation parameters $\epsilon_{\mu\nu}$.

Acknowledgments

We acknowledge discussions with Tomas Ortín, Alfonso Ramallo, Laura Tamasia and Toine Van Proeyen. We acknowledge partial financial support from projects FP2007-66665, 2009SGR502, CPAN(Consolider CSD2007-00042).

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